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Derivatives of Faber Polynomials and Markov Inequalities

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We study asymptotic behavior of the derivatives of Faber polynomials on a set with corners at the boundary. Our results have applications to the questions of sharpness of Markov inequalities for such sets. In particular, the found asymptotics are related to a general Markov-type inequality of Pommerenke and the associated conjecture of Erdős. We also prove a new bound for Faber polynomials on piecewise smooth domains. © 2002 Elsevier Science (USA)

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1. FABER POLYNOMIALS AND THEIR DERIVATIVES

Let K be a compact connected set. Denote the unbounded connected component of $\bar{\mathbb{C}}\setminus K$ by Ω. Consider the canonical conformal mapping $\Psi : \Delta \to \Omega$, where $\Delta := \{w : |w| > 1\}$, with the Laurent expansion at ∞

$$\Psi(w) = cw + c_0 + \frac{c_1}{w} + \frac{c_2}{w^2} + \cdots, \qquad |w| > 1, \quad c > 0. \tag{1.1}$$

We note that $c = cap(K)$ is the logarithmic capacity of K. The Faber polynomials $\{F_n(z)\}_{n=0}^{\infty}$, $\deg F_n = n$, are defined via the Laurent expansion of the generating function (cf. [21] or [6])

$$\frac{\Psi'(w)}{\Psi(w) - z} = \sum_{n=0}^{\infty} \frac{F_n(z)}{w^{n+1}}, \qquad z \in K, \quad |w| > 1. \tag{1.2}$$

They proved to be of considerable importance in approximation theory (see, e.g., [6, 20]), complex function theory [2] and orthogonal polynomials (cf. [22, 20]).

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An equivalent definition of Faber polynomials can be given by using the inverse conformal mapping $\Phi := \Psi^{-1}$. Then $F_n(z)$ is the polynomial part of the Laurent expansion of $\Phi^n(z)$ near $z = \infty$, i.e.,

$$\Phi^n(z) = F_n(z) + E_n(z), \qquad z \in \Omega, \tag{1.3}$$

where

$$E_n(z) = O\left(\frac{1}{z}\right) \quad \text{as } z \to \infty.$$

If the boundary of Ω is sufficiently smooth, then it is possible to show that

$$\lim_{n\to\infty} E_n(z) = 0,$$

for $z \in \Omega$, and even for $z \in \partial\Omega$ (see [21, Chap. 4; 20]). Thus, we arrive at the classical asymptotics for Faber polynomials

$$F_n(z) = \Phi^n(z) + o(1), \qquad n \to \infty, \tag{1.4}$$

where $z \in \bar{\Omega}$. Note that Faber polynomials typically tend to zero outside $\bar{\Omega}$, as $n \to \infty$ (cf. [21, Chap. 4; 7]). Using standard methods, one can prove the following asymptotics for the derivatives of Faber polynomials.

Proposition 1.1. *Suppose that $\partial\Omega$ is an analytic curve, so that Φ can be continued conformally through $\partial\Omega$. Then there exist a domain $\tilde{\Omega} \supset \bar{\Omega}$ and $r \in (0, 1)$ such that*

$$F_n^{(k)}(z) = \frac{d^k}{dz^k}(\Phi^n(z)) + O(r^n) \qquad \text{as } n \to \infty, \tag{1.5}$$

for any $z \in \tilde{\Omega}$ and $k = 0, 1, 2, \ldots$.

These asymptotics may be viewed as the differentiated versions of Eqs. (1.3) and (1.4). One can obtain a similar result, for the derivatives up to a certain order, in the case of sufficiently smooth (not analytic) boundary $\partial\Omega$. The ideas are close to those of [21, Chap. 4], but they require a much more technical argument than the proof of Proposition 1.1.

Asymptotics for Faber polynomials in the case of non-smooth boundary were obtained in [16]. If $\partial\Omega$ has the angle of opening $\alpha\pi$ at $z \in \partial\Omega$, $0 < \alpha \leqslant 2$, with respect to Ω, then (1.4) must be replaced by

$$F_n(z) = \alpha\Phi^n(z) + o(1) \qquad \text{as } n \to \infty \tag{1.6}$$

(see [16, Theorem 1.1] for the precise statement).

The primary goal of this note is to find the asymptotics for the derivatives of Faber polynomials at the corner points of $\partial\Omega$. We also consider applications of such asymptotics to Markov-type inequalities for derivatives of polynomials on K.

It is not unexpected that our subject is directly related to the geometric properties of $\partial\Omega$ via the conformal mapping Ψ. Let $z_0 \in \partial\Omega$ be a point such that two analytic arcs of $\partial\Omega$ meet at z_0 and form the angle $\alpha\pi$, $0<\alpha\leqslant 2$, as measured in Ω. According to the result of Lehman [10], $\Psi(w)$ allows an asymptotic expansion in the neighborhood of w_0, where $\Psi(w_0) = z_0$,

$$\Psi(w) - \Psi(w_0)$$

$$= \begin{cases} \sum_{k=0}^{\infty} \sum_{l=1}^{\infty} a_{kl}(w-w_0)^{k+l\alpha}, & \alpha \text{ is irrational,} \\ \sum_{k=0}^{\infty} \sum_{l=1}^{q} \sum_{m=0}^{[k/p]} a_{klm}(w-w_0)^{k+lp/q}(\log(w-w_0))^m, & \\ \quad \alpha = p/q \text{ is rational.} & \end{cases} \tag{1.7}$$

In both cases, the first term of this expansion is given by

$$\Psi(w) - \Psi(w_0) = a_\alpha(w-w_0)^\alpha + \cdots, \quad a_\alpha \neq 0 \tag{1.8}$$

(see [10, Theorem 1; 15, Sect. 3.4] for details). Our main result below gives the asymptotics for the derivatives of Faber polynomials at an "analytic corner."

THEOREM 1.1. *Let $\partial\Omega$ be rectifiable. Suppose that Ω has the angle $\alpha\pi, 0 <\alpha\leqslant 2$, at its boundary point $z_0 = \Psi(\omega_0)$, which is locally formed by two analytic arcs of $\partial\Omega$. Then*

$$F_n^{(k)}(z_0) = \frac{\alpha k! n^{\alpha k} w_0^n}{(a_\alpha w_0^\alpha)^k \Gamma(\alpha k+1)} + o(n^{\alpha k}) \qquad \text{as } n \to \infty, \tag{1.9}$$

where $k = 0, 1, 2, \ldots$.

Note that the appropriate branch of the multiple valued function w^α, $0 <\alpha\leqslant 2$, is defined by expansion (1.7)–(1.8), together with the associated coefficient a_α.

If $k=0$ then we obtain asymptotics (1.6) for Faber polynomials themselves (see [16] for a more general result). The case $k=1$ gives the asymptotics for the first derivative of Faber polynomials, which have applications to Markov-type inequalities for the derivative of polynomials on general sets. The fact that Faber polynomials can be used to show sharpness of Markov-type inequalities was already observed in the classical paper of Szegő [23]. We develop his ideas and relate our asymptotics to the result of Pommerenke [12] and the conjectures of Erdős [4, 5].

2. MARKOV INEQUALITIES FOR GENERAL SETS

Define the uniform (sup) norm on K by

$$||f||_K := \sup_{z\in K} |f(z)|.$$

The classical Markov inequality for $K = [-1, 1]$ states that

$$||P_n'||_{[-1,1]} \leqslant n^2 ||P_n||_{[-1,1]}, \tag{2.1}$$

where P_n is a polynomial of degree at most n (cf. [1, Sect. 5.1; 19]). We have equality in (2.1) for the Chebyshev polynomial $T_n(x) = \cos(n \arccos x)$. On the other hand, Bernstein's inequality for the unit disk D gives

$$||P_n'||_D \leqslant n||P_n||_D. \tag{2.2}$$

Obviously, equality holds in (2.2) for $P_n(z) = z^n$. Szegő [23] was apparently the first to explain the nature of difference in the exponents of n in (2.1) and (2.2), using the geometry of sets $[-1, 1]$ and D in the complex plane. He proved that

$$||P_n'||_K \leqslant C(K)n^{\alpha} ||P_n||_K, \tag{2.3}$$

where $\alpha\pi$ is the largest angle at $\partial\Omega$, $1 \leqslant \alpha \leqslant 2$, and $C(K)$ is independent of $n \in \mathbf{N}$. The exponent α is sharp, as shown by Szegő with the help of Faber polynomials. This also follows from Theorems 1.1 and 2.1, for $k = 1$, which in addition give a lower bound for the constant $C(K)$. Similarly, asymptotics (1.9) can be used to show the sharpness of inequalities for the derivatives of higher order $k \geqslant 2$.

A universal Markov-type inequality, for an arbitrary continuum K of capacity $cap(K)$, was obtained by Pommerenke [12]:

$$||P_n'||_K \leqslant \frac{en^2}{2\,cap(K)} ||P_n||_K. \tag{2.4}$$

Erdős conjectured that e could be replaced by 1 in (2.4) so that (2.1) would follow from this general result, as $cap([-1, 1]) = \frac{1}{2}$. After Rassias *et al.* [18]

had noticed that his conjecture needed adjustment, Erdős restated it in the corrected form

$$||P_n'||_K \leqslant \frac{(1+o(1))n^2}{2\,cap(K)}\,||P_n||_K \tag{2.5}$$

as $n \to \infty$ (see, e.g., [5]).

Note that if the angle at z_0 is 2π, in the setting of Theorem 1.1, then we have

$$|F_n'(z_0)| = \frac{1+o(1)}{|a_2|}n^2 \qquad \text{as } n \to \infty. \tag{2.6}$$

It is also known that

$$||F_n||_K \leqslant 2, \qquad n \in \mathbf{N}, \tag{2.7}$$

for convex K (cf. [14]), so that we can estimate in this case

$$\frac{||F_n'||_K}{||F_n||_K} \geqslant \frac{|F_n'(z_0)|}{2} = \frac{1+o(1)}{2|a_2|}n^2 \qquad \text{as } n \to \infty\,. \tag{2.8}$$

Thus, one might try to disprove (2.5) by finding an appropriate set K, such that $|a_2| < cap(K)$. However, we verified for a number of special cases that

$$|a_2| \geqslant cap(K). \tag{2.9}$$

In particular, we have $a_2 = 1/2 = cap([-1,1])$ for $K = [-1,1]$. After the initial version of this paper had been submitted for publication, Kühnau [9] found an elegant proof of (2.9), which is based on a distortion theorem of Löwner [11]. Hence, (2.8) and (2.9) show that inequality (2.5) is sharp for any sets with outward pointing cusps.

We remark that the convexity of K is not essential in the above argument, because (2.7) can be replaced by the following.

THEOREM 2.1. *If $\partial\Omega$ is a piecewise smooth Jordan curve formed by a finite number of Dini-smooth arcs, then*

$$\limsup_{n\to\infty}\, ||F_n||_K \leqslant 2. \tag{2.10}$$

A Dini-smooth arc is a Jordan arc with a natural parametrization $z(s)$, such that $z'(s)$ is Dini-continuous, and $z'(s) \neq 0$ for any $s \in [0,l]$ (see, e.g., [15]). Note that the bound 2 in (2.10) cannot be decreased, which is immediate from (1.6) (or from (1.9) with $k = 0$).

3. PROOFS

Proof of Proposition 1.1. Let Ω_r be a domain such that $\Phi(\Omega_r) = \{w : |w| > r\}$, $r > 0$. There exists $r_0 \in (0, 1)$ such that Φ has a conformal extension into Ω_{r_0}. Hence (1.3) is valid for any $z \in \Omega_{r_0}$, and $E_n(z)$ is analytic in Ω_{r_0}. Denote the level curve of Φ by $\gamma_r := \{z\colon |\Phi(z)| = r\}$, $r > r_0$. Using Cauchy integral formula, we obtain from (1.3) that

$$E_n(z) = \frac{1}{2\pi i} \int_{\gamma_r} \frac{\Phi^n(t)\,dt}{t - z}, \qquad z \in \Omega_r, \quad r > r_0,$$

where integration is carried in clockwise direction. It follows by differentiation of (1.3) that

$$F_n^{(k)}(z) = \frac{d^k}{dz^k}(\Phi^n(z)) + \frac{k!}{2\pi i} \int_{\gamma_r} \frac{\Phi^n(t)\,dt}{(t - z)^{k+1}}, \quad z \in \Omega_r, \quad k = 0, 1, 2, \ldots. \tag{3.1}$$

We can estimate the remainder term for $z \in \Omega_{r'}$, $r > r' < 1$,

$$\left| \frac{k!}{2\pi i} \int_{\gamma_r} \frac{\Phi^n(t)\,\mathrm{d}t}{(t - z)^{k+1}} \right| \leqslant \frac{k!}{2\pi} \frac{l(\gamma_r) r^n}{(dist(\gamma_r, \gamma_{r'}))^{k+1}}, \tag{3.2}$$

where $l(\gamma_r)$ is the length of γ_r and

$$dist(\gamma_r, \gamma_{r'}) := \min\{|t - z| : t \in \gamma_r,\ z \in \gamma_{r'}\}.$$

Thus (1.5) is a consequence of (3.1) and (3.2). ∎

Proof of Theorem 1.1. Using Cauchy formula in (1.3), for a contour $\gamma_r := \{z : |\Phi(z)| = r > 1\}$ and a point $z \in \Omega$ inside γ_r, we have that

$$F_n(z) = \frac{1}{2\pi i} \int_{\gamma_r} \frac{\Phi^n(t)\,dt}{t - z}. \tag{3.3}$$

This well-known integral representation of Faber polynomials is valid for any $z \in K$ by analytic continuation (cf. [21]). Thus, we obtain from (3.3) that

$$F_n^{(k)}(z) = \frac{k!}{2\pi i} \int_{\gamma_r} \frac{\Phi^n(t)\,dt}{(t - z)^{k+1}} = \frac{k!}{2\pi i} \int_{|w|=r} \frac{w^n \Psi'(w)dw}{(\Psi(w) - z)^{k+1}}, \tag{3.4}$$

where $z \in K$ and $k = 0, 1, 2, \ldots$. Since $\partial\Omega$ is rectifiable, $|\Psi'(w)|$ is integrable over $|w| = 1$. Therefore (3.4) gives that

$$F_n^{(k)}(z_0) = \frac{k!}{2\pi i} \int_{\gamma} \frac{w^n \Psi'(w)dw}{(\Psi(w) - \Psi(w_0))^{k+1}}, \qquad z_0 = \Psi(w_0), \tag{3.5}$$

where γ is the contour consisting of the arc $\gamma' := \{w : |w - w_0| = s,\ |w| > 1\}$ and the arc $\gamma'' := \{w : |w - w_0| \geqslant s,\ |w| = 1\}$, for a small but fixed $s > 0$. Using expansion (1.7)–(1.8), we have that (see [10, 15; Sect. 3.4])

$$\Psi(w) - \Psi(w_0) = a_\alpha (w - w_0)^\alpha + g(w - w_0)$$

and

$$\Psi'(w) = \alpha a_\alpha (w - w_0)^{\alpha - 1} + g'(w - w_0),$$

for w in a neighborhood of w_0, $|w| > 1$. The expansion for g starts as follows:

$$g(w - w_0) = \begin{cases} b(w - w_0)^{2\alpha} + \cdots, & \alpha < 1, \\ b(w - w_0)^2 \log(w - w_0) + \cdots, & \alpha = 1, \\ b(w - w_0)^{1+\alpha} + \cdots, & \alpha > 1. \end{cases}$$

Hence

$$\begin{aligned} \frac{\Psi'(w)}{(\Psi(w) - \Psi(w_0))^{k+1}} &= \frac{\alpha}{a_\alpha^k (w - w_0)^{\alpha k + 1}} + O\left(\frac{1}{(w - w_0)^p}\right) \\ &= \frac{\alpha}{a_\alpha^k w^{\alpha k + 1} (1 - w_0/w)^{\alpha k + 1}} + O\left(\frac{1}{(w - w_0)^p}\right) \\ &= \frac{\alpha}{a_\alpha^k w_0^{\alpha k + 1}} \frac{1}{(1 - w_0/w)^{\alpha k + 1}} + O\left(\frac{1}{(w - w_0)^p}\right), \end{aligned} \tag{3.6}$$

where $p < \alpha k + 1$. It follows that

$$\frac{k!}{2\pi i} \int_\gamma \frac{w^n \Psi'(w)\, dw}{(\Psi(w) - \Psi(w_0))^{k+1}} = \frac{k!}{2\pi i} \left(\int_{\gamma'} + \int_{\gamma''} \right) \frac{w^n \Psi'(w)\, dw}{(\Psi(w) - \Psi(w_0))^{k+1}}, \tag{3.7}$$

where the integral over γ'' is bounded for all $n \in \mathbf{N}$, as $s \leqslant |w - w_0| \leqslant 2$ and $|w| = 1$. Since $1/(1 - w_0/w)^{\alpha k + 1}$ is analytic in $\bar{\mathbf{C}} \setminus [0, w_0]$, we have that

$$\left| \frac{1}{2\pi i} \int_{\gamma'} \frac{w^n\, dw}{(1 - w_0/w)^{\alpha k + 1}} - \frac{1}{2\pi i} \int_{|w| = r} \frac{w^n\, dw}{(1 - w_0/w)^{\alpha k + 1}} \right| \leqslant C(s), \tag{3.8}$$

where $C(s)$ is independent of $n \in \mathbf{N}$. Using the formula for the $(n+1)$th coefficient of the Laurent expansion for $1/(1 - w_0/w)^{\alpha k + 1}$ about $w = \infty$, we obtain that

$$\frac{1}{2\pi i}\int_{|w|=r}\frac{w^n dw}{(1-w_0/w)^{\alpha k+1}} = \binom{\alpha k+n+1}{n+1} w_0^{n+1}$$

$$\sim \frac{n^{\alpha k}}{\Gamma(\alpha k+1)} w_0^{n+1} \qquad \text{as } n\to\infty. \tag{3.9}$$

The same argument shows that

$$\frac{1}{2\pi i}\int_{|w|=r}\frac{w^n dw}{(1-w_0/w)^{p}} = O(n^{p-1}) = o(n^{\alpha k}) \qquad \text{as } n\to\infty.$$

Thus, we obtain from (3.6) to (3.9) that

$$F_n^{(k)}(z_0) = \frac{\alpha k! n^{\alpha k} w_0^n}{(a_\alpha w_0^\alpha)^k \Gamma(\alpha k+1)} + o(n^{\alpha k}) \qquad \text{as } n\to\infty,$$

where $k=0,1,2,\ldots$. One can deduce more precise information about the error term, by applying similar analysis to the remaining terms of the asymptotic expansion (3.6) ∎

Proof of Theorem 2.1. Observe that Ψ extends to a homeomorphism between $\{w : |w|=1\}$ and $\partial\Omega$ (see [15, Theorem 2.1]). Consider the function

$$v(t,\theta) := \arg(\Psi(e^{it}) - \Psi(e^{i\theta})), \qquad t\neq\theta. \tag{3.10}$$

Note that $v(t,\theta)$ has a jump discontinuity as a function of t, at $t=\theta$, where $\theta\in[0,2\pi)$ is fixed. The magnitude of this jump, arising when t passes through θ, is equal to the angle formed by $\partial\Omega$ at $\Psi(e^{i\theta})$, as measured in Ω. Clearly, $v(t,\theta)$ can be defined continuously for $t\neq\theta$. It was proved in [7, Theorem 4] that $v(t,\theta)$ is of bounded variation as a function of $t\in[0,2\pi)$. Hence, we have the following integral representation for Faber polynomials:

$$F_n(\Psi(e^{i\theta})) = \frac{1}{\pi}\int_0^{2\pi} e^{int} d_t v(t,\theta), \qquad 0\leqslant\theta<2\pi, \tag{3.11}$$

which is due to Pommerenke (cf. [7, 13, 14]).

Let $\delta>0$ be small. Since $\partial\Omega$ is rectifiable, we have that $\Psi'(e^{it})\in L^1([0,2\pi))$, see [15, Theorem 6.8]. Thus (3.10) gives that

$$\int_{\theta+\delta}^{\theta+2\pi-\delta} e^{int} d_t v(t,\theta) = \int_{\theta+\delta}^{\theta+2\pi-\delta} e^{int}\Re\left(\frac{e^{it}\Psi'(e^{it})}{\Psi(e^{it})-\Psi(e^{i\theta})}\right) \mathrm{d}t. \tag{3.12}$$

The regular modulus of continuity for a 2π-periodic continuous function f is given by

$$\omega_\infty(f,u) := \sup_{|x-y|\leqslant u} |f(y)-f(x)|.$$

We also define the L^1 modulus of continuity for a 2π-periodic function $f \in L^1([0,2\pi))$ by

$$\omega_1(f,u) := \sup_{|h|\leqslant u} \int_0^{2\pi} |f(x+h)-f(x)|\,dx.$$

The corresponding L^1 modulus of continuity on $[\theta+\delta,\ \theta+2\pi-\delta]$ is denoted by $\omega_1(f,u;\theta)$. Note that

$$\min_{t\in[\theta+\delta/2,\theta+2\pi-\delta/2]} |\Psi(e^{it})-\Psi(e^{i\theta})| = c(\delta) > 0.$$

Hence, we have for $u \in (0,\delta/2)$

$$\begin{aligned}
&\omega_1\left(\Re\left(\frac{e^{it}\Psi'(e^{it})}{\Psi(e^{it})-\Psi(e^{i\theta})}\right),u;\theta\right)\\
&\leqslant \omega_1\left(\frac{e^{it}\Psi'(e^{it})}{\Psi(e^{it})-\Psi(e^{i\theta})},u;\theta\right)\\
&\leqslant \frac{\omega_1(e^{it}\Psi'(e^{it}),u)\max_{t\in[0,2\pi]}|\Psi(e^{it})-\Psi(e^{i\theta})|}{(c(\delta))^2}\\
&\quad + \frac{\omega_\infty(\Psi(e^{it}),u)\int_0^{2\pi}|e^{it}\Psi'(e^{it})|\,dt}{(c(\delta))^2}\\
&\leqslant \frac{A\omega_1(\Psi'(e^{it}),u)+\omega_\infty(\Psi(e^{it}),u)\int_0^{2\pi}|\Psi'(e^{it})|dt}{(c(\delta))^2},
\end{aligned} \tag{3.13}$$

where A is a positive constant independent of $\theta \in [0,2\pi)$ and $\delta > 0$. It follows from [3, Sect. 2.3.7] and (3.12) that

$$\int_{\theta+\delta}^{\theta+2\pi-\delta} e^{int} d_t v(t,\theta) \to 0 \qquad \text{as } n\to\infty, \tag{3.14}$$

uniformly in $\theta \in [0,2\pi)$, by a version of the Riemann–Lebesgue lemma.

We show in Lemma 3.1 that for any $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\int_{\theta-\delta}^{\theta+\delta} |d_t v(t,\theta)| \leqslant 2\pi + \varepsilon, \qquad \theta \in [0, 2\pi). \tag{3.15}$$

Combining (3.14), (3.15) and (3.11), we obtain that

$$\limsup_{n\to\infty} \|F_n\|_K \leqslant 2 + \frac{\varepsilon}{\pi},$$

which yields (2.10) after letting $\varepsilon \to 0$. ∎

Lemma 3.1. *Suppose that the assumptions of Theorem* 2.1 *are satisfied. For any* $\varepsilon > 0$ *there exists* $\delta > 0$ *such that*

$$\int_{\theta-\delta}^{\theta+\delta} |d_t v(t,\theta)| \leqslant 2\pi + \varepsilon, \qquad \theta \in [0, 2\pi).$$

Proof. We first note that the above integral expresses the variation of the angle for the secant line through $\Psi(e^{i\theta})$ and $\Psi(e^{it})$, as t runs from $\theta - \delta$ to $\theta + \delta$. This variation is clearly independent of parametrization for the arc

$$\gamma := \{\Psi(e^{it}) : \theta - \delta \leqslant t \leqslant \theta + \delta\}.$$

Also, it is well known that variation is an additive function, so that

$$\begin{aligned} Var_t(v(t,\theta), [\theta-\delta,\ \theta+\delta]) &= Var_t(v(t,\theta), [\theta-\delta, \theta)) \\ &\quad + Var_t(v(t,\theta), (\theta, \theta+\delta]) + \beta(\theta), \end{aligned} \tag{3.16}$$

where $\beta(\theta)$ is the angle at $\Psi(e^{i\theta})$ as measured in Ω. By choosing $\delta > 0$ sufficiently small, we can assume that γ contains at most one corner point of $\partial\Omega$. If γ is smooth, then $\beta(\theta) = \pi$. Furthermore, for any $\varepsilon > 0$ there is $\delta > 0$, independent of θ, such that

$$\max(Var_t(v(t,\theta), [\theta-\delta, \theta)), Var_t(v(t,\theta), (\theta, \theta+\delta])) \leqslant \varepsilon/2,$$

by Theorem 5 of [7]. This gives that

$$Var_t(v(t,\theta), [\theta-\delta, \theta+\delta]) \leqslant \pi + \varepsilon, \tag{3.17}$$

uniformly in θ.

If $\Psi(e^{i\theta})$ is a corner point, then we similarly obtain that

$$Var_t(v(t,\theta), [\theta-\delta, \theta+\delta]) \leqslant \beta(\theta) + \varepsilon \leqslant 2\pi + \varepsilon. \tag{3.18}$$

Consider the remaining case when the corner point is at $\Psi(e^{it_0})$, $t_0 \in (\theta, \theta+\delta)$. Following the same argument as for (3.17), we still have that

$$Var_t(v(t,\theta), [\theta-\delta, t_0]) \leqslant \pi + \varepsilon/2, \tag{3.19}$$

for all sufficiently small $\delta > 0$, which are independent of θ. Thus, we need to estimate $Var_t(v(t,\theta), [t_0, \theta+\delta])$. Note that the point $\Psi(e^{i\theta})$ is located outside the arc

$$\gamma_1 := \{\Psi(e^{it}) : t_0 \leqslant t \leqslant \theta+\delta\},$$

but it can be arbitrarily close to γ_1. We now consider a more general variation function

$$h(z) := Var(\arg(\zeta - z), \zeta \in \gamma_1), \qquad z \in \bar{\mathbf{C}}.$$

Let $\zeta_j := \Psi(t_j)$, $j = 0, \ldots, k$, where $t_0 < t_1 < \cdots < t_k = \theta + \delta$, be a partition of γ_1. Observe that

$$h_k(z) := \sum_{j=0}^{k-1} |\arg(\zeta_j - z) - \arg(\zeta_{j+1} - z)|$$

is a continuous subharmonic function on $\bar{\mathbf{C}} \backslash \gamma_1$, for any $k \in \mathbf{N}$. By the (generalized) maximum principle for subharmonic functions (cf. [17, Theorems 2.3.1 and 3.6.9]), we have that

$$h_k(z) \leqslant \max_{\xi \in \gamma_1} h_k(\xi) \leqslant \max_{\xi \in \gamma_1} h(\xi), \qquad z \in \bar{\mathbf{C}} \backslash \gamma_1.$$

Letting $k \to \infty$, we obtain that

$$h(z) \leqslant \max_{\xi \in \gamma_1} h(\xi), \qquad z \in \bar{\mathbf{C}} \backslash \gamma_1.$$

Since ξ is now positioned on the smooth arc γ_1, it follows again that

$$Var_t(v(t,\theta), [t_0, \theta+\delta) \leqslant \max_{\xi \in \gamma_1} Var(\arg(\zeta - \xi), \zeta \in \gamma_1) \leqslant \pi + \varepsilon/2,$$

as in (3.17) and (3.19). Combining (3.19) with the above estimate, we have that

$$Var_t(v(t,\theta), [\theta-\delta, \theta+\delta]) \leqslant 2\pi + \varepsilon$$

in this remaining case too, so that the lemma is proved. ∎

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